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INCOMPATIBILITY, DEFECTS, AND STRESS FUNCTIONS IN THE MECHANICS OF GENERALIZED CONTINUA

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Abstract—Materials with internal structure are in the focus of today's interest. We investigate a particularly simple class of such materials which, nevertheless, is of great practical use; namely the Bravais crystals, i.e. (mono- or poly-) crystals with only one atom in the elementary cell. Plastic deformation causes incompatible elastic strain and leads to an internal stress state. The incompatibility, explained as some misfit of matter inside the body, is exclusively due to so-called crystallographic defects, if nonmechanical effects such as thermal and magnetostrictive strains are excluded. The elementary defects are classified as point, line, and interface defects. They can be described in the language of field theory if the *continuized* Bravais crystal is introduced by a limiting procedure, as explained. It is argued that any space described by affine differential geometry is isomorphic to the continuized Bravais crystal. More complex crystals must be treated by more complex geometries.

Point defects are described as nonmetric objects, line defects (dislocations) as torsion, and interface defects as nonconnective objects of the material space representing the crystal. The concept of stress space is introduced. This space is dual to the strain space considered so far. The stress function tensor (or tensor potential) is the metric tensor of the stress space. The other basic quantities describe the response to the presence of the defects in the form of force stresses and of (symmetric and antisymmetric) double-force stresses.

The given equations provide a frame within which all phenomena involving crystallographic defects have to fit. For Bravais crystals with such defects, they play a role, analogous to that of Maxwell's equations in electromagnetism.

1. INTRODUCTION

It has been found that conventional continuum mechanics is not rich enough to describe many important phenomena, as, for instance, plasticity associated with materials in practical use. Therefore, continuum mechanics has been extended in various directions, for example by the introduction of internal variables specifying the internal mechanical state. This state is so important because the mentioned phenomena are often highly irreversible, so that a realistic description must consider the sequence of states through which the body passes. We shall argue in Section 2 that, in a purely mechanical situation, this state is built up from crystallographic defects, provided the material is crystalline. Noncrystalline materials are not considered in this article. We shall restrict ourselves to Bravais crystals because these have the narrowest variety of defects. On the other hand, they have certain basic types of defects, namely point defects, line defects, and interface defects. Therefore, the restriction to Bravais crystals does not eliminate most of what is interesting in our field.

In Part I of this work we introduce, among others, the concept of the continuized (Bravais) crystal. This crystal is a generalized continuum in which three crystallographic directions can be identified at "almost" every point. Points where they cannot be identified group themselves at single points and along lines and interfaces. This permits us to define the three types of defects mentioned above. It is argued that the continuized crystal is isomorphic to any space which is described by affine differential geometry. All defects are identified as important quantities of this geometry.

In Part II we show that the quantities which represent the response to the presence of the defects are conveniently considered in the so-called stress space which is dual to the strain space regarded so far. The duality, which is a fundamental feature of the theory, implies that the equations for the response quantities, generalized stresses, obey the equations of the affine geometry of the stress space. In this respect, the stresses are analogous to the defects rather than to the strains, as emphasized by Kleinert[1].

PART I. THE KINEMATICS OF THE INTERNAL MECHANICAL STATE

2. STATE AND STATE QUANTITIES

The physics of the solid state or, simpler, solid state physics belong to the most extended disciplines in the general area of physics. This is so because, depending on the state in which the solid is found, its properties and behaviour may vary enormously. Similarly extensive is the manifold of different states.

The mechanics of the solid state or, simpler, solid state mechanics is the mechanical part of solid state physics. The term "solid state mechanics" is not often used, although it contains an important message. In fact, it emphasizes the decisive role of the state in mechanical application. It is, perhaps, useful to recall how the notion of state is introduced in physics and mechanics. Firstly, the state quantity is defined as a quantity which is characteristic for the present situation of the physical system and can, at least in principle, i.e. by a thought experiment, be measured at any time without having information about the past. To give an example: The elastic strain is a state quantity because it can be measured in the way just explained. In fact, we may imagine that a volume element is cut out from a deforming body at any time. After relaxing the stress we measure the change in shape and volume of the elements and thus obtain the elastic strain. It is easy to show that plastic strain is not a state quantity because, for its measurement, information from the past is needed.

Physical systems are usually described by more than one state quantities. All these state quantities assume certain values at any instant of time. By these values they characterize what is called the state of the system at that time.

3. INCOMPATIBILITY

Internal state quantities are variables which have a meaning independent of outside action on the body. Thus, a solid can be in an internal mechanical state even in the absence of external action. A simple example is an elastic solid in an internal stress state, i.e. a stressed body which is not subject to any external forces. This problem was investigated by H. Reissner[2], who noticed that internal stresses in a simply connected medium are not in contradiction with Kirchhoff's theory.

An internal, or incompatible, stress state (eigen stress state) can be understood as follows. If the stressed body is cut into small elements, in which the stress is then relaxed, we obtain an assembly of elements which do not fit together. Thus, an incompatible deformation implies that a nonfitting collection of elastic elements is united to a compact body. If the nonfitting occurs on the infinitesimal scale, then we can obtain an internal stress state which varies continuously through the body.

A well-known realization of this situation occurs in the plastic deformation of a crystalline solid. Imagine that an element of the body suffers a plastic deformation, for instance by the motion of dislocations. If during this deformation the element were isolated, it would change its shape such that it would, in general, no longer fit together with the remainder of the body. If, however, the medium is to remain compact, then the element will suffer a corresponding constraint from its surroundings. This constraint causes an additional elastic strain such that the total strain, plastic and elastic, is compatible, although both plastic and elastic strains are incompatible. If this consideration is extended to all elements of the body, then the relation to the procedure of the preceding paragraph becomes obvious. In fact, imagine that the undeformed body is cut into elements which then are deformed plastically. In general, they now form a nonfitting collection which is then united to a compact body by (incompatible) elastic deformation.

After the described deformation, the medium is in an internal mechanical stress state characterized by the presence of dislocations. These can be seen and their arrangement measured, e.g. by electron microscopy. Since it is not necessary for such measurements to know anything about the past, we conclude that any quantity describing dislocation arrangements is a (mechanical) state quantity.

4. DEFECTS AS INTERNAL MECHANICAL STATE VARIABLES

It is a fundamental, perhaps surprising, feature of this theory that it is not the incompatibility, but rather the dislocation, which is the primary state quantity. By the term "primary" we would like to stress the following: If some position-dependent quantity is a state variable, then its spatial derivative is also a state variable. It is shown in the theory that the incompatibility due to dislocations is obtained in the form of some derivatives of the dislocation density tensor. Thus, incompatibility is not a primary state quantity.

Incompatibility can also arise in rather different situations. Well-known are stresses due to temperature and magnetization variation. In this work, we shall restrict ourselves to purely mechanical states. Then, incompatibility is always due to crystal defects.

Dislocations are line-shaped defects. In the frequently occurring Bravais crystals which are constructed by repeated translations of a point (atom)—rather than of a cell, containing several atoms—in three nonplanar directions, the dislocation is the only elementary (as opposed to composed) line defect. Since the defect structure in Bravais crystals is much simpler than in other crystals, we shall only consider those. In real or imagined high resolution microscopy point defects, so-called vacancies and (self-) interstitials, as well as interface defects can also be seen in a space which otherwise appears empty. This suggests to us to look at the crystal in analogy to the vacuum and at the elementary defects in analogy to the elementary particles filling the vacuum. In this view, the crystal can be compared with the universe, and it is perhaps not too surprising that the equations governing the defect state are formally similar to those of the universe, in particular to those of general relativity theory.

The above considerations make sense in situations where crystallinity is the prevailing feature of the solid. This is the case when all defects can be identified distinctly and individually. This is not so in the so-called amorphous solids which, therefore, are excluded from this theory.

If we restrict ourselves to pure mechanics, then the internal state of our solid is completely specified by the defects and their distribution. Since these defects largely determine the properties and the behaviour of the solid, there exists basic interest in a theory of crystal defects. The question then arises whether a continuum theory or a field theory will fulfill the purpose. It is good to distinguish between these two types of theories. It is a fundamental feature of what is now understood as a physical field theory that the physical space in question contains individual singularities which surround themselves with fields, and in this way interact with each other. A continuum in the ordinary sense lacks these singularities and, therefore, is closer to an amorphous rather than to a crystalline solid. In the last 20 years, however, certain "generalized" continua have been investigated, among others the pseudocontinuum introduced by Rogula[3] and Kunin[4]. Whereas these generalized continua are good for some purposes, they do not seem to smooth the way for a field theory of defects. This goal, however, may be achieved by the notion of the continuized crystal, to be explained below.

5. THE CONTINUIZED CRYSTAL

Because the lattice parameter is rather small (atomic spacing $\sim 10^{-10}$ m) compared to other lengths that are important in the particular problem, we can neglect its finiteness altogether and treat the solid as a continuum. A rough way to do this is to distribute the mass of each atom over the available space such that the medium is continuously filled with mass. The medium would now be a true continuum which remembers its crystalline state only via the anisotropy of its properties. Obviously, this procedure leaves no space for individual defects.

Consider now a limiting process consisting of a sequence of steps of such a nature: In each step the three lattice constants (each one belonging to one of the mentioned nonplanar directions) shall be reduced by a constant factor, say, k (e.g. $k = \frac{1}{2}$), and at the same time, the mass of the particles shall be reduced by the factor k^3. Obviously,

this process leaves the mass density constant. Let us call the mass points in the lattices produced in the described way "quasi-atoms". No matter how far we have gone in our procedure, the following statement will always be correct, as long as the crystal does not contain defects: Three crystallographic directions can be recognized at each individual quasi-atom. Thus, the main characterization of a crystal, namely the existence of crystallographic directions with some length scale, is preserved in the limiting procedure. After N steps, the lattice constants are reduced by factor of k^N. This means that we have the impression of seeing the original crystal if we look at the "N-step crystal" with a magnification of k^{-N}.

So far, we have considered the so-called ideal crystal, which is the undeformed and undefected crystal. The crystal which is elastically deformed but defect-free shall be named "perfect." Thus, the ideal crystal is a special case of the perfect crystal. To study the perfect crystal, consider a volume element, say ΔV. If ΔV is small enough, e.g. infinitesimal, then its strain state is homogeneous plus a nonhomogeneous strain which is small of order $(\Delta V)^2$. Hence, the strain state is homogeneous in the limit $\Delta V \to 0$, and the magnification statement applies as before.

Now we come to the defected crystal, which can be taken as undeformed from outside, according to the last result. Simplest is the case of point defects. These become "quasi-point defects" and are reduced in strength analogous to the quasi-atoms. The magnified N-step crystal again has the appearance of the original crystal, with point defects.

The last result is a consequence of the fact that the density of point defects, like that of mass, is measured per unit volume. Dislocation densities are measured per unit area. If we insist that the dislocation density, defined as the resulting Burgers vector of all dislocations intersecting an area element, is preserved in the continuization procedure, then we find that the distance between neighbouring dislocations, when measured in quasi-atomic spacings, increases like k^{-1} in each step. Thus, the magnification statement is not valid for dislocations. Of course, the distance between the quasi-dislocations decreases in our procedure when it is measured with a constant scale.

Interface defect densities are measured per unit length of intersecting lines. They are not well explored. Therefore we omit a deeper discussion of these.

6. DIFFERENTIAL GEOMETRY OF DEFECTS

The particular signification of the continuized crystal lies in the fact that the (quasi-) atomic configurations of this crystal are exactly described by the so-called affine differential geometry.

Metricity

Consider first the perfect crystal. Lengths are measured and atoms identified by counting lattice steps in the three crystallographic directions, then applying Pythagoras' theorem

$$ds^2 = g_{kl}\, dx^k\, dx^l, \tag{1}$$

where ds is the distance of two atoms with relative position dx^k. It is convenient to use cartesian coordinates. ds as introduced in eqn (1) is not the distance obtained by an external observer by means of a constant scale, but is, rather, the distance found by an internal observer with the help of the counting procedure. The first named distance, when squared, would be $\delta_{kl}\, dx^k\, dx;^l$. The difference of the two distances is, of course, the "external" strain in the perfect crystal.

If dislocations are introduced, then the counting procedure remains practicable. Uncertainties, which occur because it makes a difference of one step when a dislocation is once passed on the right- and then on the left-hand side, are small of higher infinitesimal order and need not be regarded. The uncertainty is, nevertheless, a fundamental

feature of the dislocation, because it is directly related to Frank's Burgers circuit, by which the dislocation is defined in crystal physics. A continuized version of this circuit has been introduced into differential geometry by Cartan before the discovery of the crystal dislocation. It was the great discovery of Kondo[5] and of Bilby, Bullough, and Smith[6] that Cartan's torsion, a fundamental notion of differential geometry, and dislocation are essentially the same thing. We shall soon introduce a quantity Γ^k_{ml}, called the affine connection of the space considered in our differential geometry. We mention in passing that the torsion is quantitatively described by that part of Γ^k_{ml} which is antisymmetric in the subscripts m, l. Details of this are described in many texts on differential geometry, so that they can be omitted here (see e.g. Schouten[7]).

The situation is different with the point defects, which in the continuation procedure retain their distance. Thus, the internal observer will meet a point defect after the same number of steps in the real crystal as in the continuized crystal. Each time, he does not know what to do. This means that, unlike dislocations, point defects break the metricity. A nonmetric geometry is now needed.

Nonmetric situations are possible within affine differential geometry. There the nonmetricity is quantitatively defined and is another fundamental notion. It follows from our consideration that the nonmetricity can be used as a measure of point defect densities (Kröner[8], Zorawski[9]).

Affine differential geometry is based on the operation of parallel transport of a vector. Two vectors, say $v^k(x^m + \mathrm{d}x^m)$ and $v^k(x^m)$ are said to be parallel, or relatively parallel-displaced, with respect to a quantity Γ^k_{ml} introduced for this purpose, if

$$\mathrm{d}v^k + \Gamma^k_{ml} v^l \, \mathrm{d}x^m = 0. \tag{2}$$

Γ^k_{ml} is a central quantity of affine differential geometry and is called the "affine connection," also the linear connection or affinity. In fact, it defines the connectivity of the considered space. Vectors (and also tensors) can also be displaced parallel along a finite path. If this path is closed, one obtains

$$\oint_C \mathrm{d}v^k = -\oint_C \Gamma^k_{ml} v^l \, \mathrm{d}x^m = -\tfrac{1}{2} \iint^S R^k_{nml} v^l \, \mathrm{d}S^{nm}, \tag{3}$$

where S is the area bounded by the circuit C and

$$R^k_{nml} \equiv 2(\partial_n \Gamma^k_{ml} - \Gamma^k_{np} \Gamma^p_{ml})_{[nm]} \tag{4}$$

is the curvature tensor, which vanishes if, and only if, the transported vector coincides with the original vector. If the curvature tensor vanishes for arbitrary circuits, the situation is that of teleparallelism.

If any two points of our space can be connected by paths along which parallel transport is possible, then the space is said to be connected, or to possess connectivity. It is important for us that admissible paths for parallel transport cannot pass through elementary interface defects. This statement can even be used for the definition of these defects. Thus interface defects break the connectivity. That means that the affine connection is no longer well-defined, and the curvature tensor, which now is to be introduced independently of the connection, does not obey the Bianchi identities which arise from the particular form (4) of the curvature tensor. The "nonfulfillment" can be made quantitative and named "nonconnectivity." Our result is, then, that the nonconnectivity can be used as a measure of interface defect densities.

The details of nonmetric and nonconnective defects are not yet well explored. It seems as if the affine differential geometry with the extension to nonconnective situations has just enough freedom to describe completely the (quasi-) atomic configurations of the continuized defected crystal. If this is true, then any space described by affine differential geometry is isomorphic to a continuized crystal.

II. THE STATICS OF THE INTERNAL MECHANICAL STATE

7. LINEAR ELASTICITY THEORY

Consider, first, states caused by external loads on the surface and in the interior of the body. In the linearized theory, the relation between elastic strain, $\underline{\epsilon}$, and displacement, $\mathbf{u}$, is

$$\epsilon_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i) \quad \text{or} \quad \underline{\epsilon} = \operatorname{def} \mathbf{u} \tag{5}$$

(for the operator "def" read "deformation of"). The special form (5) of the strain tensor implies the compatibility equations

$$E^{ij} \equiv -e^{ikm}e^{jln}\partial_k\partial_l\epsilon_{mn} = 0, \quad \text{or} \quad \underline{E} \equiv \operatorname{inc} \underline{\epsilon} = 0, \; E^{ij} \equiv E^{ji} \tag{6}$$

(for the operator "inc" read "incompatibility of"). The operator identities

$$\operatorname{inc} \operatorname{def} \equiv 0, \qquad \operatorname{div} \operatorname{inc} \equiv 0 \tag{7}$$

are easily proved. They play a similar role in the theory with symmetric tensor fields as do the well-known identities

$$\operatorname{curl} \operatorname{grad} \equiv 0, \qquad \operatorname{div} \operatorname{curl} \equiv 0 \tag{8}$$

in a theory with vector fields. In particular, any symmetric tensor field, say $\underline{\tau}$, can be decomposed according to

$$\underline{\tau} = \operatorname{def} \mathbf{a} + \operatorname{inc} \underline{\phi}, \tag{9}$$

where $\mathbf{a}$ and $\underline{\phi}$ are a suitably chosen vector field and symmetric tensor field respectively.

Here, we are interested in the *internal* stress situation. We, therefore, set the external forces equal to zero and are left with the following set of equations:

$$\operatorname{div} \underline{\sigma} = 0, \quad \sigma_{ij} = \sigma_{ji}, \quad \underline{\epsilon} = \underline{\underline{s}} \ldots \underline{\sigma}, \quad \underline{E}(\equiv \operatorname{inc} \underline{\epsilon}) = \underline{\eta}. \tag{10}$$

For simplicity, we consider situations in which boundary conditions need not be taken into account. Note that eqn (5) can no longer be used as a definition of the strain, because an elastic displacement field does not exist in the incompatible situations. Therefore, the strain is to be defined by means of eqn (1), for instance as

$$\epsilon_{kl} = \tfrac{1}{2}(g_{kl} - \delta_{kl}), \tag{11}$$

if cartesian coordinates are chosen. This definition is well-known in the nonlinear theory of elasticity. The third eqn (10) is the inverse of the linear law of elasticity, valid also in the case of elastic anisotropy. In the fourth eqn (10) $\underline{\eta}$ is the symmetric tensor field of incompatibility, which is a quantitative measure of the nonfitting of the elements described in Section 3. Note that from the definition of $\underline{E}$

$$\operatorname{div} \underline{E} \equiv 0, \qquad E_{ij} \equiv E_{ji}, \tag{12}$$

hence

$$\operatorname{div} \underline{\eta} \equiv 0, \qquad \eta_{ij} \equiv \eta_{ji}. \tag{13}$$

The first two eqns (10) are satisfied identically by

$$\underline{\sigma} = \operatorname{inc} \chi, \qquad \chi_{ij} = \chi_{ji}, \tag{14}$$

where $\underline{\chi}$ is known as the Beltrami stress function tensor (also stress potential or tensor potential). A combination of eqns (10) and (14) leads to

$$\mathrm{inc}(\underline{\underline{s}} \ldots \mathrm{inc}\ \underline{\chi}) = \underline{\eta} \tag{15}$$

as the set of simultaneous differential equations for $\underline{\chi}$. Since a stress potential of the form def a does not contribute to $\underline{\sigma}$, an extra condition can be imposed on $\underline{\chi}$. An interesting choice, for the case of elastic isotropy, can be formulated in terms of an auxiliary potential

$$\chi^1_{ij} = \frac{1}{2G}\left(\chi_{ij} - \frac{\nu}{1+2\nu}\,\delta_{ij}\chi\right), \tag{16}$$

where G is the shear modulus and ν is Poisson's ratio. The condition, given first by Kröner[10] and Marguerre[11], reads

$$\mathrm{div}\ \underline{\chi}^1 = 0. \tag{17}$$

Using this "gauging", eqn (15) reduces to

$$\nabla^4 \underline{\chi}^1 = \underline{\eta} \tag{18}$$

which is in accord with eqns (13). The anisotropic case has been treated by Kröner[12], but will not be reviewed here. This reference contains a review of the state of our field in 1980.

The simplicity of eqns (17) and (18) makes the stress potential $\underline{\chi}'$ (or $\underline{\chi}$) a most useful tool for incompatible strain problems. The reader who is familiar with electrodynamics notices the analogy between the there-introduced vector potential and the tensor potential $\underline{\chi}$. We mention in passing that the dynamic problem of stress functions was treated, e.g. by Kluge[13].

8. NONLINEAR ELASTICITY THEORY

In the compatible situation, the strain can be related to the displacements in the manner described in the textbooks on nonlinear elasticity theory. The compatibility equations are then expressed as the vanishing of a tensor, which is the nonlinear analogon to the tensor $\underline{E}$. For simplicity, also this tensor will be called $\underline{E}$. If, in the differential expression defining $\underline{E}$, the ϵ_{kl} are replaced by g_{kl}, according to eqn (11), then it has exactly the form of the 3-dimensional Einstein tensor (recall that the 4-dimensional Einstein tensor occurs in general relativity theory). In three dimensions, the Einstein tensor is related to the Riemann-Christoffel curvature tensor, another fundamental quantity of differential geometry, by

$$K_{nmlk} = e_{nmi}e_{lkj}E^{ij}. \tag{19}$$

This tensor has to be distinguished from the curvature tensor R_{nmlk} introduced in Section 6. In fact, $\underline{\underline{K}}$ is a special case which arises when the affine connection defining $\underline{\underline{R}}$ has the special form of a Christoffel symbol. This is the case of Riemannian geometry. These remarks also explain the name "Riemann-Christoffel" tensor.

The (nonlinear) compatibility equations are equivalent to the vanishing of the Riemann-Christoffel curvature. This result has been known for a long time. It was quoted, for instance,by Trefftz[14].

In the incompatible situation, the Einstein tensor and the Riemann-Christoffel tensor no longer vanish. Of course, the definition of $\underline{E}$, i.e. the nonlinear generalization of the first part of eqns (6), remains valid. The identities corresponding to those of eqns

(12) are now

$$\nabla_i E^{ij} = 0, \qquad E^{ij} = E^{ji}, \tag{20}$$

with ∇_i as the symbol of covariant differentiation in the Riemannian space considered here. $\nabla_i E^{ij}$ is the divergence of $\underline{E}$ in this space. In differential geometry, the two equations in (20) are often called the first and second Bianchi identities. Using eqn (19), they can also be written in terms of the Riemann-Christoffel tensor.

In linear approximation, the first two eqns (10) have the same form as eqns (20). Overmore, the relation between $\underline{\sigma}$ and $\underline{\chi}$ is formally equal to that between $\underline{E}$ and $\underline{g}$. Thus, we can interpret the first two eqns (10) as linearized Bianchi identities of a space with metric tensor $\underline{\chi}$. This was first proposed by Schaefer[15], who also stressed the relation to relativity theory. We shall now postulate that the force and moment equilibrium conditions for the internal stress state are the nonlinear Bianchi identities

$$\nabla_i^* \sigma^{ij} = 0, \qquad \sigma^{ij} = \sigma^{ji}, \tag{21}$$

where ∇_i^* is the symbol of covariant differentiation in the stress space, thus different from ∇_i. The strongest argument for this postulate is that it makes the stress space dual to the strain space and vice versa. In fact, the equations of the stress space are now exactly of the same form as those of the strain space. The deeper reason of this duality stems from the fundamental duality of general mechanics, i.e. from the duality of space and momentum, as it manifests itself, for instance, in Hamilton's canonical equations. A final proof for the postulate does not yet exist. For recent work on the problem of duality, see Amari[16].

9. THE STRESS SPACE OF THE CONTINUIZED CRYSTAL WITH DEFECTS

In Part II of the article, we have not yet spoken about the origin of the incompatibilities. Assume now that these come from the defects discussed in Part I. Since the space defined by the Riemann-Christoffel tensor is metric and connective, it is too narrow to account for point and interface defects. We therefore introduce the more general curvature tensor R^k_{nml} and the corresponding space. In view of the duality of stress and strain space, we expect that the Bianchi identities of the stress space are the static field equations of the mechanical defect state. These equations are equilibrium equations for the force stresses, and for the generalized stresses that arise as response to the presence of dislocations and point defects. These stresses are double-force stresses with and without moment. The equilibrium equations now have a more general form than eqns (21). For instance, it was shown by Stojanović[17] and Kröner [18] that, in the absence of point defects, but in the presence of dislocations, the second of eqns (21) are the well-known equilibrium equations for moments, if moment stresses are admitted. One finds that these moment stresses form the torsion of the stress space and are the specific response to dislocations.

Similarly, the specific response to point defects corresponds to the nonmetricity of the stress space. These are generalized stresses of the type "double force without moment." A more detailed study of these stresses is desirable.

In connection with the interface defects, we had mentioned the possibility that the space could be nonconnected. In that case, the Bianchi identities are no longer valid. For a stress space, such a situation would mean that external forces and double forces act upon our body. This is most easily seen on hand in the first eqn (10), if there the zero is replaced by the external force density. Also, the last results demand deeper studies. One may altogether say that the static part of the theory is much less explored than the geometric-kinematic part.

10. CONCLUSION

We have shown that the elementary crystallographic defects, as well as the pertaining response quantities, are well described by the affine differential geometry of two mutually dual spaces named strain and stress space. A most helpful concept for this is the concept of the continuized crystal. Whereas certain details of the theory still need clarification, it seems that the general picture is correct. In particular, we believe that the continuized Bravais crystal represents that generalized continuum, which is most appropriate for the description of Bravais crystals. Note that many important materials such as iron, copper, nickel, and aluminum are of the Bravais type. The utility of the concept of the continuized Bravais crystal does not at all diminish the value of other concepts of generalized continua. Such concepts are extremely important, for instance for the theory of composite materials.

The theory developed here is not closed because, so far, we have not connected the stress space with the strain space. The missing part is, therefore, some unification of the two spaces, usually based upon the constitutive equations. Because they contain all the irreversibility of the processes with defects, we expect that such equations which contain the whole information about the material must be extremely complex. The constitutive equations used so far have always had a limited range of applications, and this will probably remain so.

The equations of differential geometry are basically nonlinear. Of course, they can be linearized, but at a cost of missing the fundamental phenomena which cannot be described within a linear theory. Important in this respect is the mutual conversion of defects as well as their creation and annihilation; e.g. dislocations create point defects in so-called nonconservative motion.

We have omitted completely the dynamics of defects as well as the so-called gauge theories which represent another approach to establish the theory of the internal mechanical state (defect state). Dynamical equations are obtained by adding inertia terms to statical equations. It seems that the main difficulties with dynamics do not lie so much in the equations of differential geometry extended to dynamics, but rather in the constitutive equations which were outside the scope of this article.

The gauge theory of defects has become fashionable in recent times, mainly due to the great success of gauge theories in other fields of physics. The first proposal to apply gauge theoretical concepts to defects seems to have occurred in little noticed work by Turski in 1967[19]. Alicia Golebiewska-Herrmann, to whom this issue and this article is dedicated, drew early attention to the analogies between dislocations and other field theories. She proposed to apply gauge theories and had, herself, first successes[20–22]. Since then, many works in this field have followed (for references see, e.g. the report[23]). It is doubtful whether the gauge theory will lead us to fundamental new results that cannot be obtained in another way, for instance with the methods of this article. The great value of gauge theories is rather their unifying conception, applicable to all field theories. This conception makes optimum use of symmetry and is, therefore, most precious for the understanding of the underlying physics.

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